

C24 Dynamical Systems

Lecture 6: Limit Cycles

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Michaelmas Term 2025

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Lecture 6 overview

- This lecture will focus on analyzing **limit cycles**, conditions for their existence and stability
- Last lecture the Poincaré-Bendixson theorem gave us criteria to establish whether closed orbits exist; we can also establish if they do not exist, through **Bendixson's** and **Dulac's criteria**
- **Index theory** will help us characterize closed trajectories in the phase plane, and to determine whether it is possible for orbiting trajectories to exist
- We will analyze stability of limit cycles using the concept of a **Poincaré map**

Periodicity

Definition: A solution $\phi(\mathbf{x}_0, t)$ of the autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ satisfying $\phi(\mathbf{x}_0, 0) = \mathbf{x}_0$ is called **periodic** if there exists some $T > 0$ such that

$$\phi(t, \mathbf{x}_0) = \phi(t + T, \mathbf{x}_0) \text{ for all } t \in \mathbb{R}$$

- Given a periodic solution $\phi(\mathbf{x}_0, t)$, the minimal value of $T > 0$ for which $\phi(t, \mathbf{x}_0) = \phi(t + T, \mathbf{x}_0)$ is called the **period** of the solution

- This lecture only considers orbits with finite period T

(We exclude separatrix cycles because it takes infinite time for homoclinic or heteroclinic orbits to go from α to ω limits, and $t + T$ then makes no sense)

Proving a periodic orbit does not exist

Bendixson's criterion can be used to show that a given 2nd order dynamical system does not have any periodic solutions

Let $\mathbf{x} = (x, y) \in \mathbb{R}^2$ denote the state of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and recall that

$$\operatorname{div}(\mathbf{f}) = \nabla \cdot \mathbf{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} \quad \text{if } \mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_x(x, y) \\ f_y(x, y) \end{bmatrix}$$

Bendixson's criterion: if $\nabla \cdot \mathbf{f}$ is not identically zero, and if $\nabla \cdot \mathbf{f}$ does not change sign in a simply connected region D of the phase plane, then the 2nd order system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has no closed orbits in D

Outline proof for Bendixson's criterion

- Since x and y are parametric in t , the solution trajectories satisfy

$$\begin{aligned} \dot{x} &= f_x \\ \dot{y} &= f_y \end{aligned} \implies \frac{dy}{dt} / \frac{dx}{dt} = \frac{f_y}{f_x} \implies \frac{dy}{dx} = \frac{f_y}{f_x}$$

- Suppose a closed orbit $\Gamma \subset D$ exists, then $f_x dy - f_y dx = 0$ on Γ , so

$$\oint_{\Gamma} (f_x dy - f_y dx) = 0$$

and by Stokes' theorem, if Γ encloses a region $S \subset D$ then

$$\oint_{\Gamma} (f_x dy - f_y dx) = \int_S (\nabla \cdot \mathbf{f}) dx dy = \int_S \left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} \right) dx dy = 0$$

- So If $\nabla \cdot \mathbf{f}$ is nonzero and doesn't change sign in D , then our supposition must be false, i.e. no closed orbit is possible

Modification: Dulac's criterion

Consider the same differential equations, but also allow the functions $f_x(x, y)$ and $f_y(x, y)$ to be multiplied by another function $B(x, y)$

Dulac's criterion: if B is a continuously differentiable function on a domain D of the phase plane, and if the quantity

$$\frac{\partial(Bf_x)}{\partial x} + \frac{\partial(Bf_y)}{\partial y} = \nabla \cdot (B\mathbf{f})$$

is not identically zero and does not change sign in the domain, then the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has no closed orbits in the domain D

Bendixson example 1

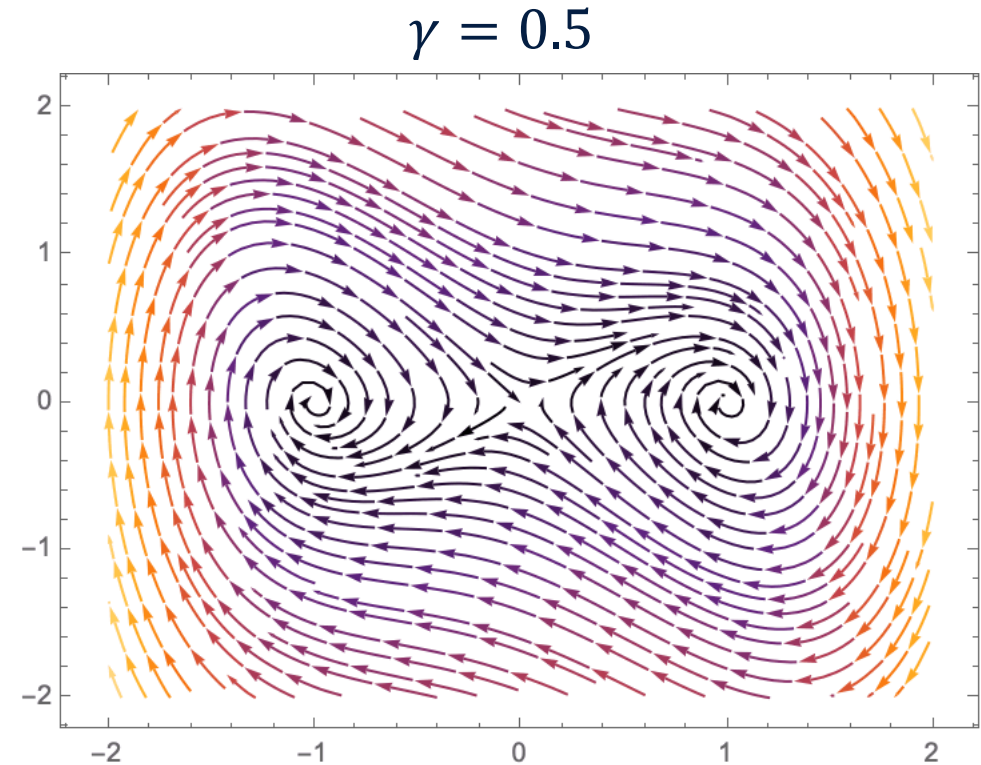
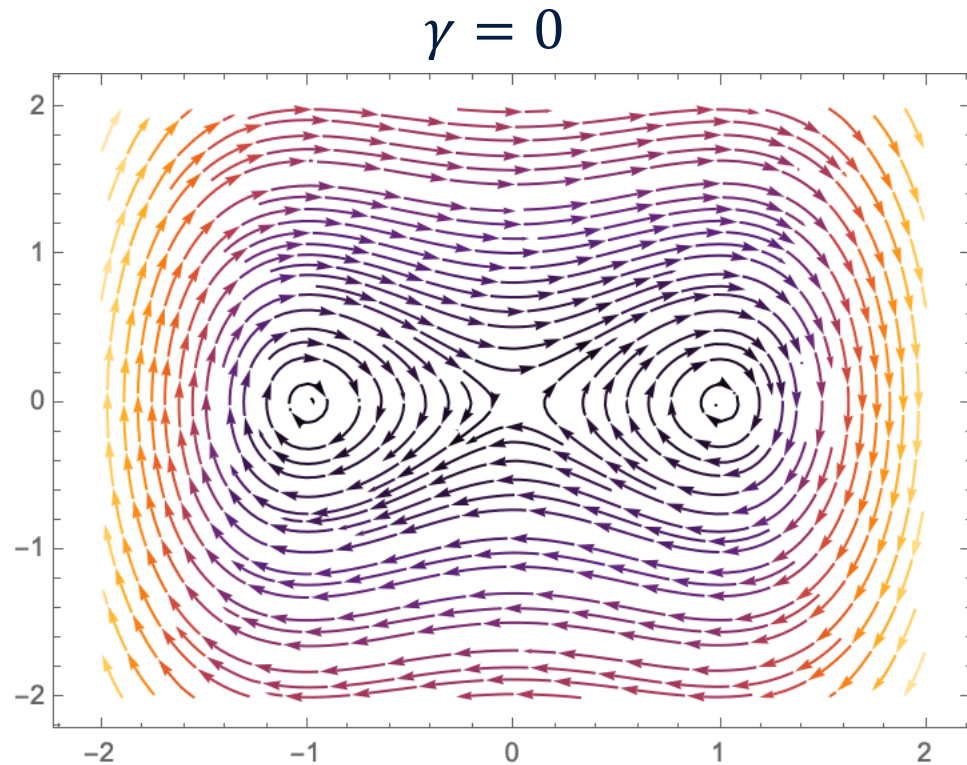
Return again to the Duffing oscillator, which is described for $\gamma \geq 0$ by

$$\begin{aligned}\frac{dx}{dt} &= y & &= f_x(x, y) \\ \frac{dy}{dt} &= x - x^3 - \gamma y & &= f_y(x, y)\end{aligned}$$

- Here $\nabla \cdot \mathbf{f} = -\gamma$, so Bendixson's criterion implies that:
 - for $\gamma \neq 0$ there are no solution trajectories that are closed orbits
 - for $\gamma = 0$ periodic solutions are possible
- As we saw in lecture 4, for $\gamma = 0$ the system is Hamiltonian, and its trajectories can be studied using the level sets of the Hamiltonian function

Example 1 visualisation

- closed orbits for $\gamma = 0$
- no closed orbits for $\gamma > 0$



Bendixson example 2

Now modify the second Duffing oscillator equation to get

$$\begin{aligned}\frac{dx}{dt} &= y &= f_x(x, y) \\ \frac{dy}{dt} &= x - x^3 - \gamma y + x^2 y &= f_y(x, y)\end{aligned}$$

- Here $\nabla \cdot \mathbf{f} = -\gamma + x^2$
- Using Bendixson's criterion, it be can't concluded that there are no closed orbits
 - there can't be a closed orbit entirely within a region of phase space where $\nabla \cdot \mathbf{f} < 0$ or $\nabla \cdot \mathbf{f} > 0$
 - but orbits could exist because $\nabla \cdot \mathbf{f}$ can change sign

Gradient systems and orbits

Recall that for a gradient system we have $\dot{\mathbf{x}} = -\nabla V$

- Consider the time-derivative of the potential function:

$$\dot{V} = \frac{dV}{dt} = \nabla V \cdot \dot{\mathbf{x}} = -\dot{\mathbf{x}} \cdot \dot{\mathbf{x}} = -\|\dot{\mathbf{x}}\|^2$$

- If the solution is on a closed orbit of period T , then we must have

$$V(\mathbf{x}(T + t)) - V(\mathbf{x}(t)) = 0 \quad \forall t$$

But integrating \dot{V} w.r.t. t gives

$$V(\mathbf{x}(T + t)) - V(\mathbf{x}(t)) = \int_t^{t+T} \dot{V} dt = - \int_t^{t+T} \|\dot{\mathbf{x}}\|^2 dt$$

and the only way this can equal zero is if $\mathbf{x}(t)$ is at an equilibrium point, so **gradient systems cannot have periodic solutions**

Gradient systems and orbits

An impossible scenario

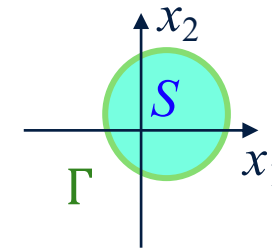


Index theory

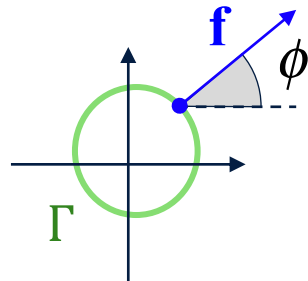
- For two-dimensional systems, we have seen that analyzing solution trajectories is facilitated by using techniques applicable to fluid flow
- Bendixson's criterion checks the circulation of a vector field:

$$\left. \begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned} \right\} \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$\oint_{\Gamma} \begin{bmatrix} -f_2 \\ f_1 \end{bmatrix} \cdot d\mathbf{l} = \int_S \nabla \times \begin{bmatrix} -f_2 \\ f_1 \end{bmatrix} \cdot d\mathbf{S} = \int_S \nabla \cdot \mathbf{f} dS$$



- Index theory translates circulation into a quantity I that takes integer values; I is the change in the angle a flow makes with the x_1 axis when traversing loop Γ

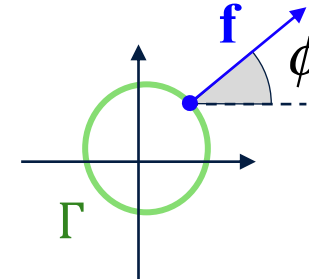


$$\phi(x_1, x_2) = \arctan\left(\frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}\right)$$

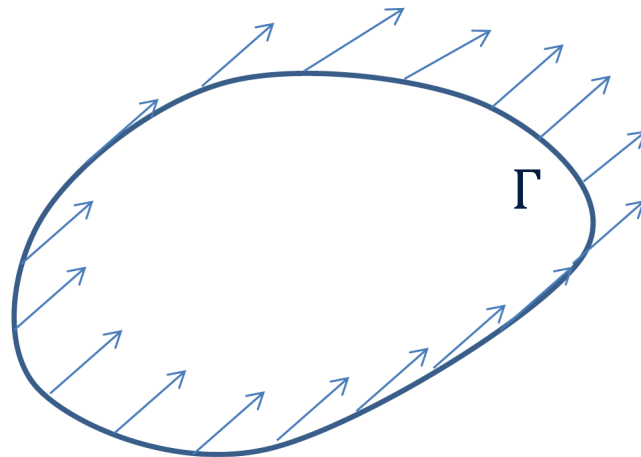
Index of a curve

- The index of a non-intersecting, continuously differentiable closed plane curve Γ (i.e. a simple loop), written $I(\Gamma)$, is defined as

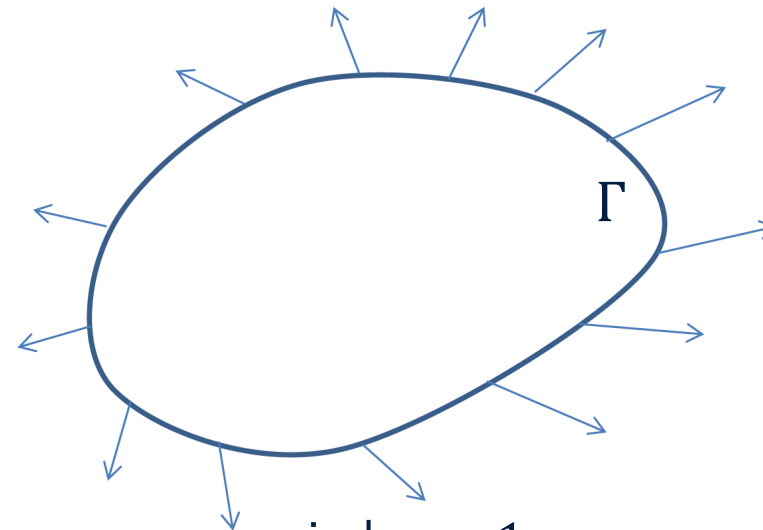
$$I(\Gamma) = \frac{1}{2\pi} \oint_{\Gamma} d\phi$$



- Qualitatively, the index measures how many times the vectors on the curve rotate anticlockwise during one anticlockwise trip around the loop



index = 0



index = 1

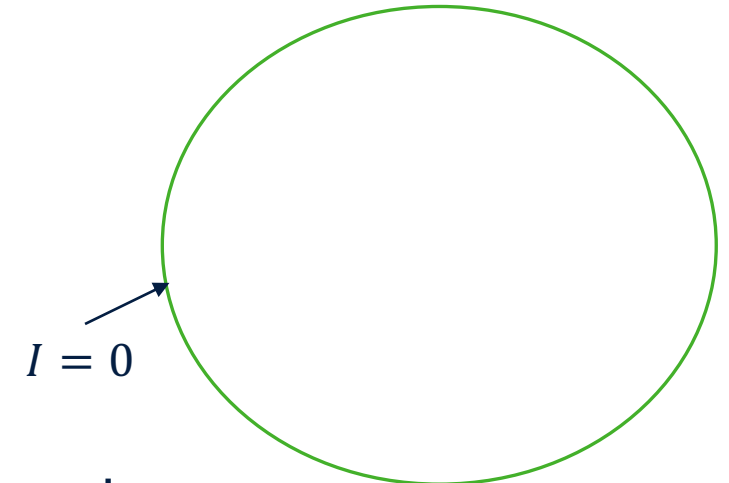
Properties of indices

- The index is always an integer (one must always rotate by a multiple of 2π to get the flow angle back to where it started)
- If there are no equilibria inside a loop Γ , then its index is $I(\Gamma) = 0$
- If loop Γ coincides with a closed orbit, then $I(\Gamma) = 1$
- If loop Γ encloses a saddle equilibrium point, then $I(\Gamma) = -1$
- If loop Γ encloses any other equilibrium point, then $I(\Gamma) = 1$
- The index of a loop that encloses multiple equilibria is the sum of the indices of loops around the individual equilibria enclosed

General conclusions from indices

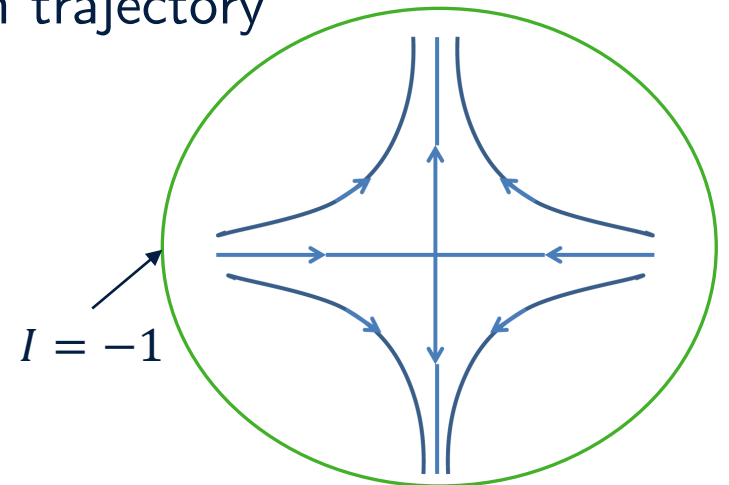
- Any loop with index $= 0$ that doesn't contain equilibrium points can't be a solution trajectory

To be a valid trajectory, it would have to be an orbit, which requires index $= 1$



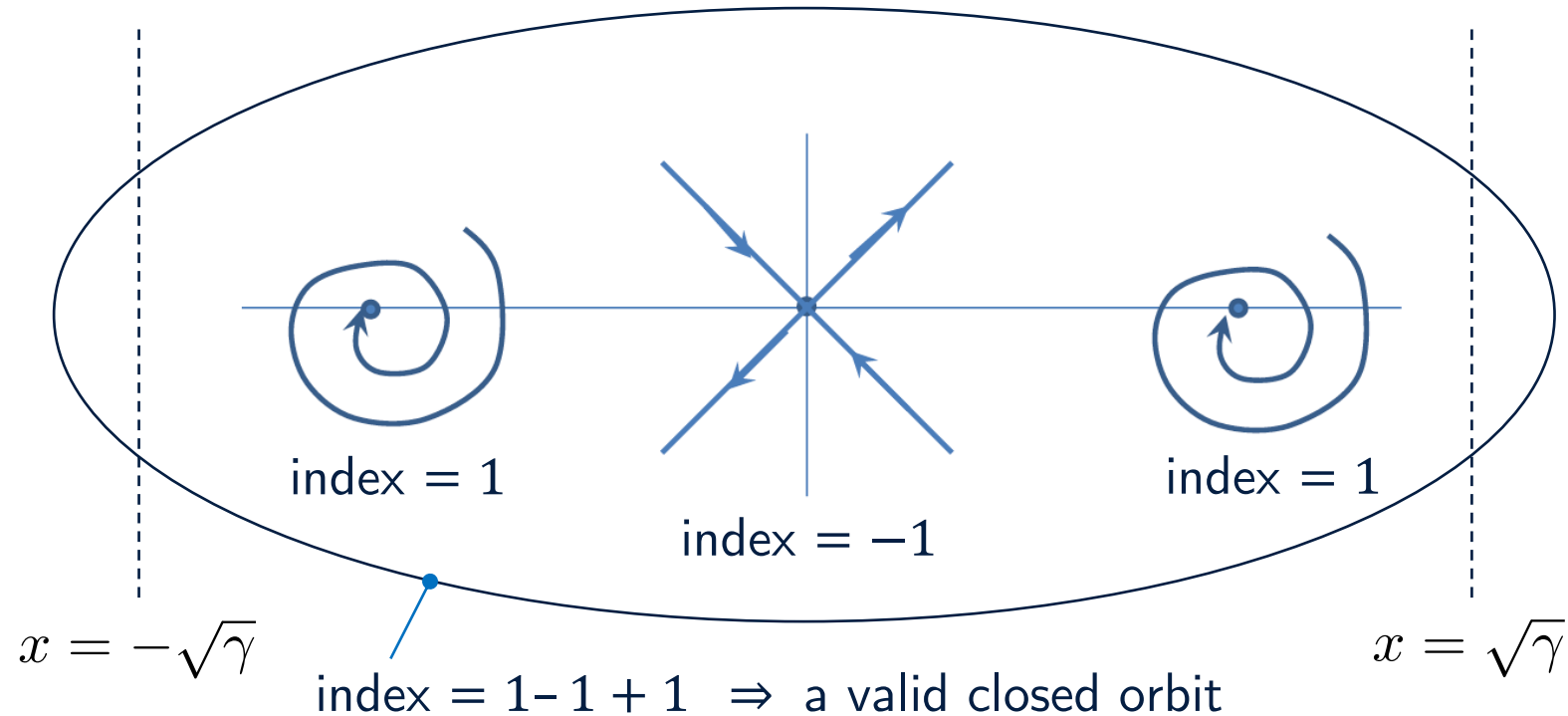
- Any loop around a single saddle node can't be a solution trajectory

To be a valid trajectory, it would have to be an orbit, so it can't have index $= -1$



Return to example 2

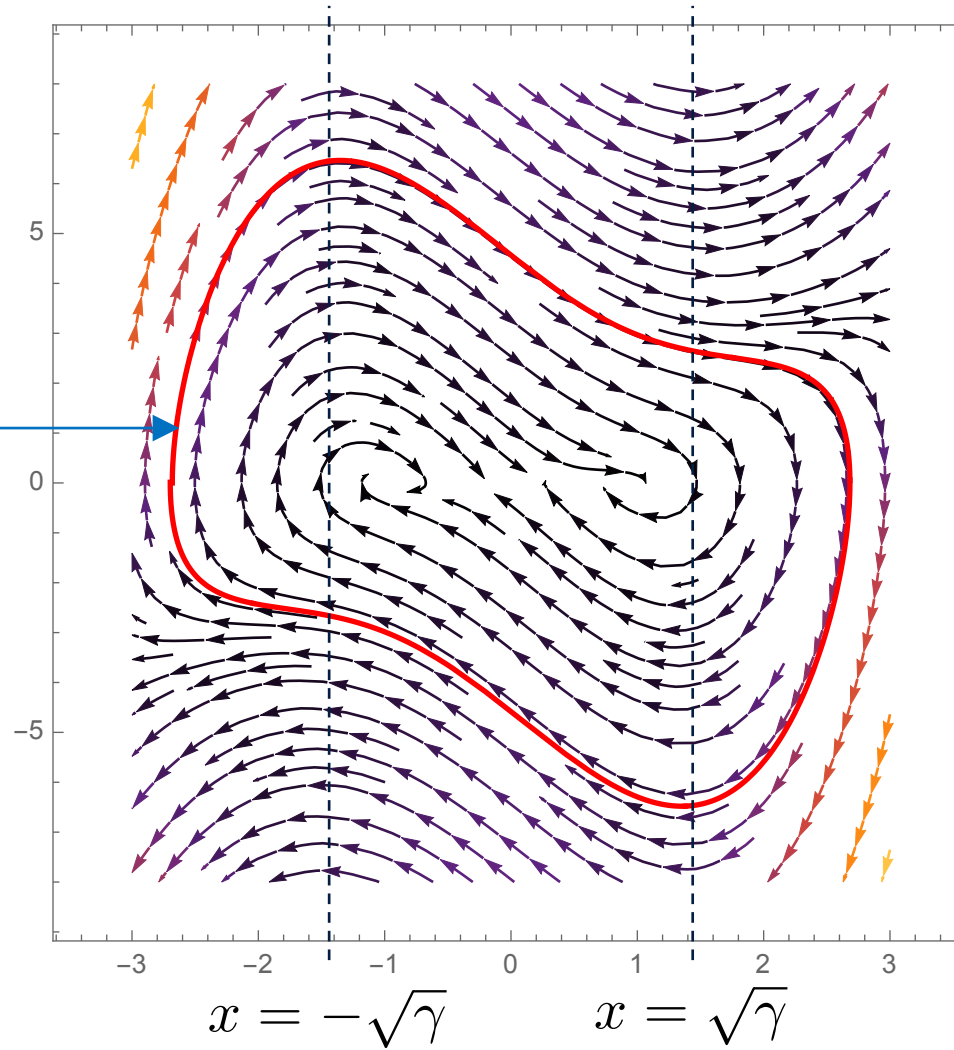
- Governing system: $\dot{x} = y$
 $\dot{y} = x - x^3 - \gamma y + x^2 y$
- Three hyperbolic equilibria: $(-1,0)$, $(0,0)$, $(1,0)$
stable nodes or foci at $(\pm 1, 0)$ and a saddle node at $(0,0)$



Example 2 visualisation

Phase portrait for $\gamma = 2$

Periodic solution
in predicted
location!



Example 3

Consider

$$\begin{aligned}\dot{x}_1 &= x_1(3 - x_1 - 2x_2) \\ \dot{x}_2 &= x_2(2 - x_1 - x_2)\end{aligned}$$

- Equilibrium points: $\mathbf{x}^* = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

- Jacobian: $D\mathbf{f}(\mathbf{x}) = \begin{bmatrix} 3 - 2x_1 - 2x_2 & -2x_1 \\ -x_2 & 2 - x_1 - 2x_2 \end{bmatrix}$

$$\det(D\mathbf{f}(\mathbf{x}^*) - \lambda I) = 0$$

$$\implies \lambda = (3, 2), (-2, -1), (-3, -1), (-1 \pm \sqrt{2})$$

- properties: unstable node, stable node, stable node, saddle
indices: 1, 1, 1, -1

Example 3

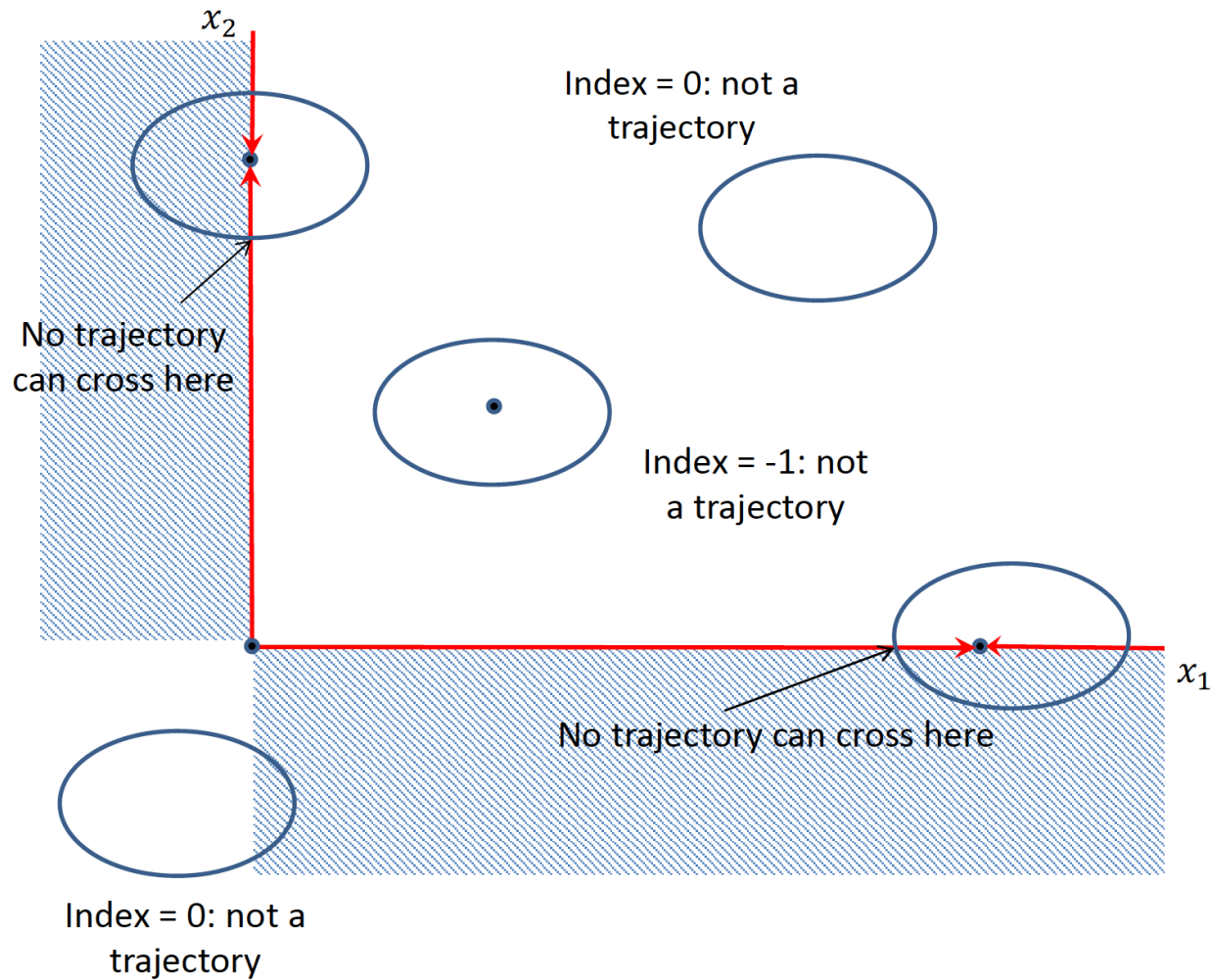
Equilibrium points \mathbf{x}^* :

$$\begin{array}{cccc} \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \\ 2 \end{bmatrix}, & \begin{bmatrix} 3 \\ 0 \end{bmatrix}, & \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ I = 1 & I = 1 & I = 1 & I = -1 \end{array}$$

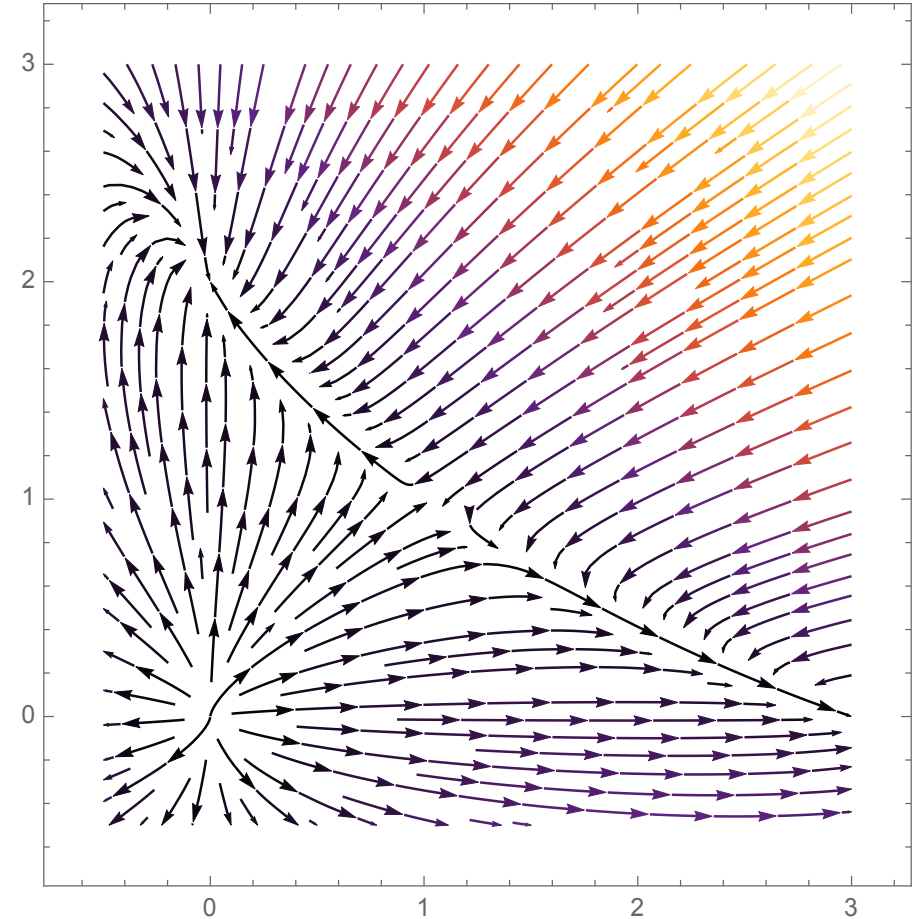
- a valid orbit must have index $I = 1$
 - trajectories cannot cross
 - no equilibria in the 2nd, 3rd, or 4th quadrants so they cannot contain a closed trajectory
 - there are trajectories lying on the x_1 and x_2 axes, so no trajectory can cross into the 2nd, 3rd, or 4th quadrants
 - since trajectories cannot encircle the equilibria that lie on the axes, it is not possible to enclose a set of indices that add to 1
- \implies there are no possible closed orbits

Example 3 visualization

- Graphical illustration of arguments



- Phase portrait



Limit cycles and stability

So far we have used the term “limit cycle” informally – it’s time for a rigorous definition

- Limit cycles are isolated periodic orbits, which can be stable or unstable (a cycle around a centre is not isolated and hence is not a limit cycle)
- In the phase plane, a limit cycle is necessarily the α or ω limit set of some trajectory other than itself

Definition: A periodic orbit Γ is said to be **stable** if for every $\epsilon > 0$ there is a neighbourhood U of Γ such that, for all $\mathbf{x} \in U$ and all t , the distance between $\phi(t, \mathbf{x})$ and the closest point in Γ is less than ϵ .

Definition: The orbit Γ is called **asymptotically stable** if it is stable and, for all $\mathbf{x} \in U$, this distance tends to zero as $t \rightarrow \infty$.

Conditions for limit-cycle stability

Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ have a periodic solution $\mathbf{x} = \gamma(t)$, $0 \leq t \leq T$, then the periodic orbit Γ lies on $\gamma(t)$. This periodic orbit is asymptotically stable only if

$$\int_0^T \nabla \cdot \mathbf{f}(\gamma(t)) dt \leq 0$$

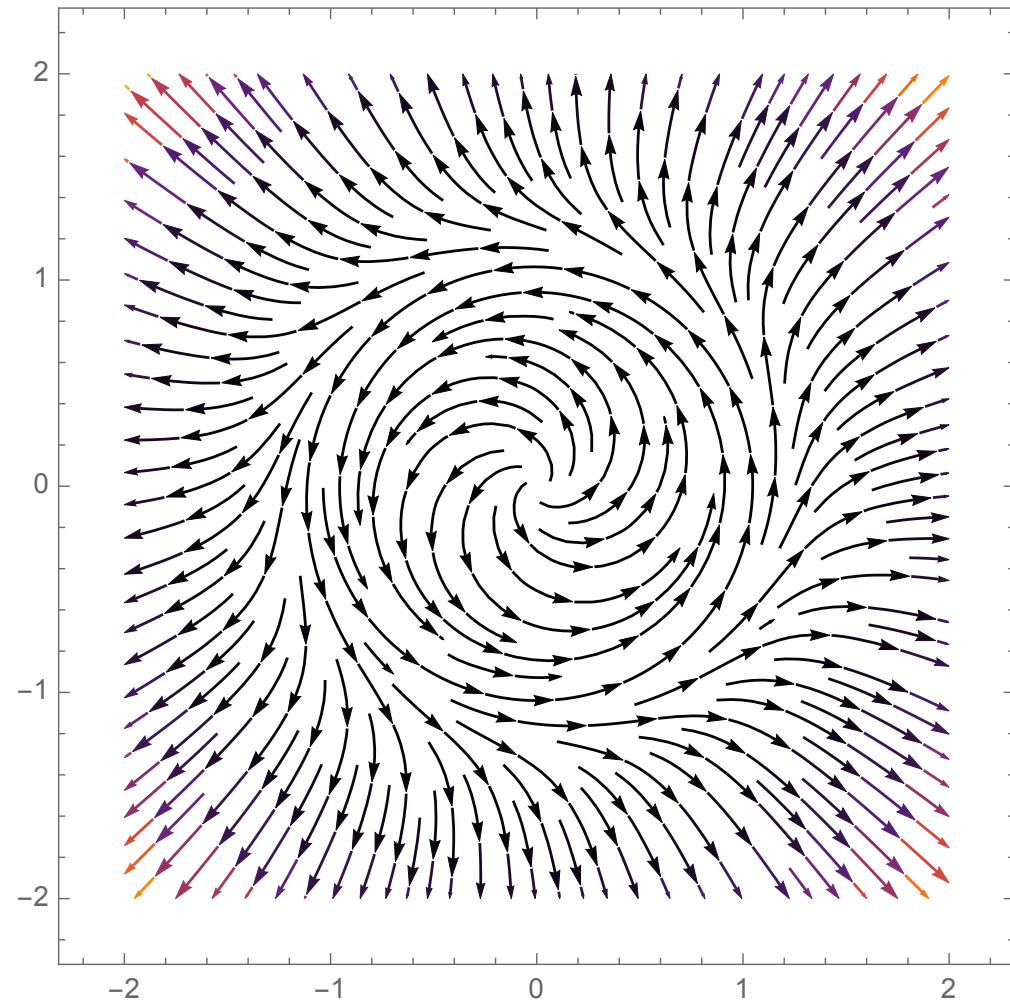
- For planar systems, if Γ is the ω limit set of all trajectories in a neighbourhood of Γ , then it is a **stable** limit cycle
- For planar systems, if Γ is the α limit set of all trajectories in a neighbourhood of Γ , then it is an **unstable** limit cycle
- For planar systems, if Γ is the ω limit set for one trajectory and the α limit set for another, it is a **semi-stable** limit cycle

Limit cycle example

Examine the autonomous system

$$\begin{aligned} \dot{x} &= -y + x(1 - x^2 - y^2)^2 & \iff & \quad \dot{r} = r(1 - r^2)^2 \\ \dot{y} &= x + y(1 - x^2 - y^2)^2 & & \quad \dot{\theta} = 1 \end{aligned}$$

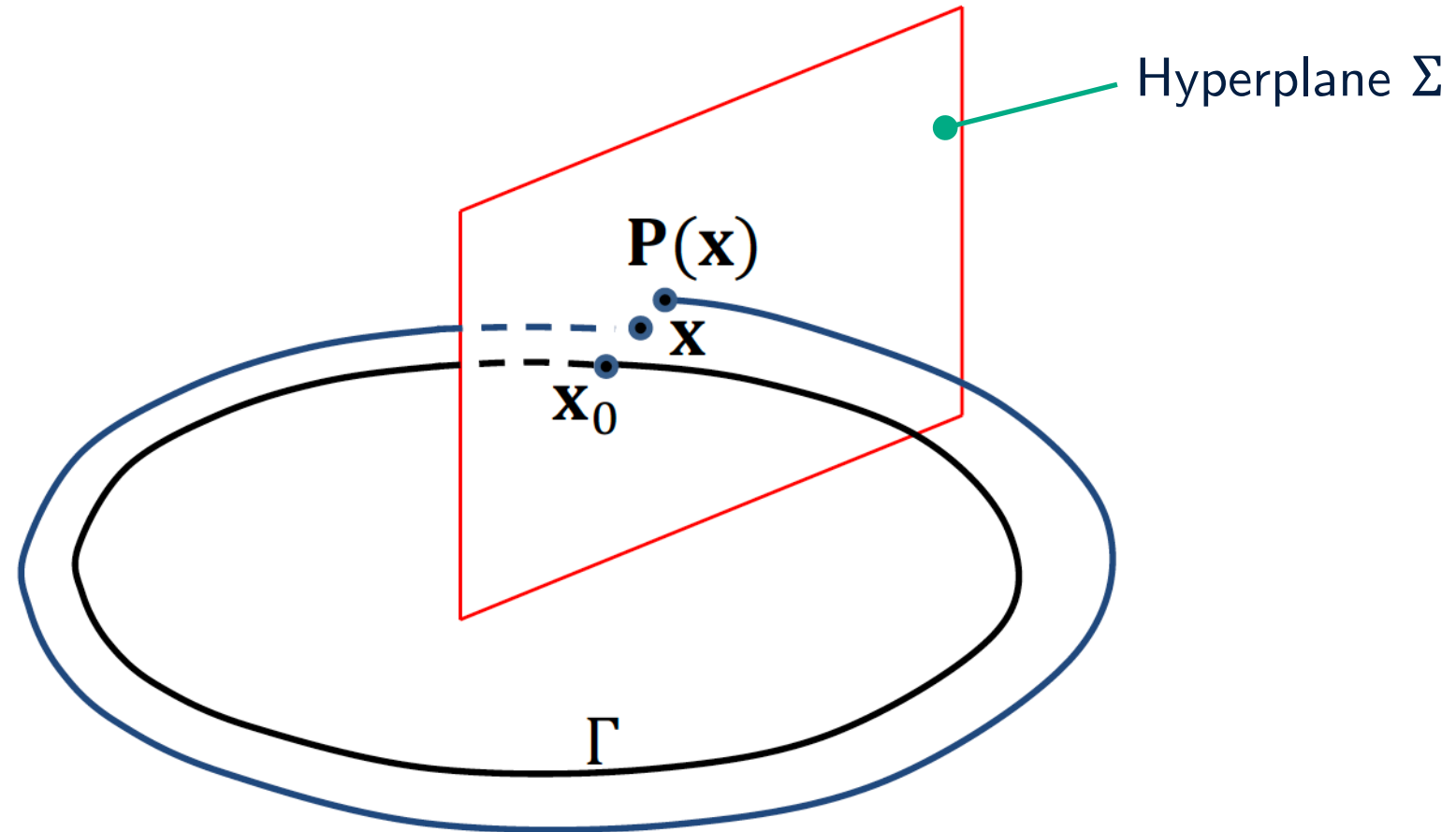
- For $r \neq 1$, $\dot{r} > 0$
therefore solution trajectories spiral outwards
- For $r = 1$, $\dot{r} = 0$
therefore a semi-stable limit cycle



The Poincaré map

- The Poincaré map (sometimes called the ‘return map’) is an important tool for the analysis of dynamical systems
- For a periodic orbit, consider a hyperplane Σ that is perpendicular to the orbit’s trajectory
- Given a point \mathbf{x} on the orbit and in the hyperplane Σ , consider where the point moves to once it has traversed the orbit once; this process defines a map $\mathbf{x} \mapsto \mathbf{P}(\mathbf{x})$
- As this mapping is iterated, the intersection point moves in the perpendicular hyperplane
- If it is a periodic orbit, then the iteration of the map will arrive at a fixed point, $\mathbf{x}_0 = \mathbf{P}(\mathbf{x}_0)$

Poincaré map graphical representation



Poincaré map example

Return to the system we discussed in Lecture 5:

$$\begin{aligned} \dot{x} &= -y + x(1 - x^2 - y^2) \\ \dot{y} &= x + y(1 - x^2 - y^2) \end{aligned} \iff \begin{aligned} \dot{r} &= r(1 - r^2) \\ \dot{\theta} &= 1 \end{aligned}$$

- In Lecture 5 we showed that this has a stable limit cycle, which is an attractor for the whole plane excluding the origin
- Solving for $(r(t), \theta(t))$ given initial condition (r_0, θ_0) :

$$\begin{aligned} \frac{dr}{dt} &= r(1 - r^2) \implies \int_{r_0}^{r(t)} \frac{dr}{r(1 - r^2)} = \int_0^t dt = t \\ \implies r(t) &= \frac{1}{\sqrt{1 - (1 - \frac{1}{r_0^2})e^{-2t}}}, \quad \theta = \theta_0 + t \end{aligned}$$

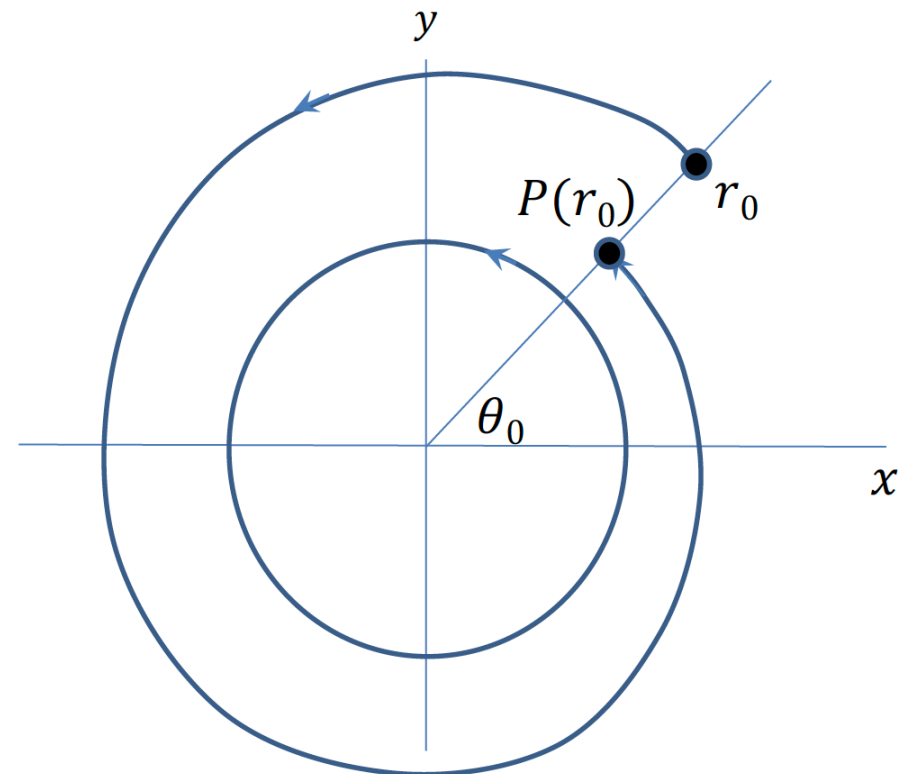
Poincaré map example continued

- Consider the hyperplane Σ defined by the ray $\theta = \theta_0$ through the origin that's crossed by a solution trajectory at times $t = 0, 2\pi, 4\pi, \dots$

$$P(r_0) = \frac{1}{\sqrt{1 - \left(\frac{1}{r_0^2} - 1\right)e^{-4\pi}}}$$

- Here $P(1) = 1$, so $r = 1$ is a fixed point
- This is a stable limit cycle because

$$\left. \frac{dP}{dr} \right|_{r=1} = e^{-4\pi} < 1$$



Questions?